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DISCUSSION OF BUCKLING STRESSES FOR FLAT PLATES AND SECTIONS

(Published in July, 1951)

By P. P. Bijlaard, and Elbridge Z. Stowell, George J. Heimerl, Charles Libove, and Eugene E. Lundquist

STRUCTURAL DIVISION

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DISCUSSION

P. P. BIJLAARD,[66] M. ASCE.—The extensive results obtained by the Structures Research Division of the National Advisory Committee for Aeronautics (NACA) are made more readily available by the publication of this paper. As a matter of course, the paper refers to applications to aluminum alloys and to aircraft structures. Similar computations referring to bridge engineering were presented by the writer[67,68,69,31,70] from 1938 to 1947.

As mentioned by the authors, the exact theory of buckling of plate assemblies[68,69,31] and that of the plastic buckling of plates[67,68,69,31] were developed by the writer. A short survey of the historical developments in this field has been given by G. Winter,[71] M. ASCE. As far as the first theory is concerned the NACA later made available very useful tables for the rotational stiffnesses S^{II}, S^{III}, and S^{IV} of flat rectangular plates under compression. These stiffnesses refer to an unloaded edge subjected to a sinusoidally varying moment, if the far edge is simply supported, free, or subjected to a moment equal and opposite to that at the near edge, respectively. These stiffnesses (or rather, specific angle rotations) were derived by the writer[68,69,31] in 1939 and 1940 and in 1943 by the NACA.[72, 27] Using the NACA derivations, they were tabulated by W. D. Kroll[26] in 1943. The symbols in the paper, referring to quarter radians, differ somewhat from those preferred by the writer. In the elastic range, for S^{II}, S^{III}, and S^{IV}, the writer prefers $N/(4\,\theta)$, in which N is the flexural rigidity of the restrained plate, and θ is θ_a, θ_c, or θ_b, as defined elsewhere.[73] The tables by Miss Kroll[26] are a very definite help in finding values θ_a, θ_b, and θ_c, for the elastic range in the writer's equations, by means of the substitutions $N/(4\ S^{II})$, $N/(4\ S^{IV})$, and $N/(4\ S^{III})$, respectively. The writer's theory takes account of the fact that the intersection lines of the middle planes of the plates do not remain straight. This is of importance in case of flexural and torsional buckling of **T**-sections and angles.[31,74]

NOTE.—This paper by Elbridge Z. Stowell, George J. Heimerl, Charles Libove, and Eugene E. Lundquist was published in July, 1951, as *Proceedings-Separate No. 77*. The numbering of footnotes in this Separate is a continuation of the consecutive numbering used in the original paper.

[66] Prof. of Structural Eng., Cornell Univ., Ithaca, N. Y.; formerly Prof., Technical Universities of Delft, Holland, and Bandoeng, Netherlands East Indies.

[67] "A Theory of Plastic Stability and Its Application to Thin Plates of Structural Steel," by P. P. Bijlaard, *Proceedings*, Royal Netherlands Academy of Sciences, Amsterdam, Holland, Vol. 41, No. 7, September, 1938, pp. 731–743.

[68] "Exacte Berekening voor het Plooien van de Wanden der in de Bruggenbouw Gebruikelijke Staafprofielen," by P. P. Bijlaard, *De Ingenieur in Nederlandsch Indie*, Vol. 6, February, 1939, pp. I.17–I.23.

[69] "Afleiding van Eenvoudige Gebruiksformules en Grafieken ter Bepaling van het Plooigevaar van de Wanden van Vloeistalen Staafprofielen," by P. P. Bijlaard, *ibid.*, October, 1939, pp. I.239–I.256.

[31] "Theory of the Plastic Stability of Thin Plates," by P. P. Bijlaard, *Publications*, International Association for Bridge and Structural Eng., Vol. 6, 1940–41, p. 45.

[70] "Some Contributions to the Theory of Elastic and Plastic Stability," by P. P. Bijlaard, *Publications*, International Assn. for Bridge and Structural Eng., Vol. 8, 1947, pp. 17–80.

[71] "Performance of Compression Plates as Parts of Structural Members," by G. Winter, *Engineering Structures Supplement*, Academic Press Inc., New York, N. Y., 1949, pp. 179–185.

[72] "Principles of Moment Distribution Applied to Stability of Structures Composed of Bars or Plates," by Eugene E. Lundquist, Elbridge Z. Stowell, and Evan H. Schuette, *Wartime Report No. L-326*, 1943.

[27] "Principles of Moment Distribution Applied to Stability of Structures Composed of Bars or Plates," by Eugene E. Lundquist, Elbridge Z. Stowell, and Evan H. Schuette, *Report No. 809*, NACA, 1945, Eq. 13.

[26] "Tables of Stiffness and Carry-over Factor for Flat Rectangular Plates Under Compression," by W. D. Kroll, *Wartime Report No. L-398*, NACA, 1943.

[73] "Theory of the Plastic Stability of Thin Plates," by P. P. Bijlaard, *Publications*, International Assn. for Bridge and Structural Eng., Vol. 6, 1940–1941, Eqs. (48, 52), (56, 57), or (53, 54).

[74] "On the Torsional and Flexural Stability of Thin Walled Open Sections," by P. P. Bijlaard, *Proceedings*, Royal Netherlands Academy of Sciences, Amsterdam, Holland, Vol. 51, March, 1948, pp. 314–321.

The writer contends[31] that, in connection with unavoidable eccentricities. for columns as well as for plates subject to buckling in the plastic range, it should be assumed that there is no unloading of initially plastically deformed regions; the plate should be assumed to remain entirely plastic.[75,76,77] However, at that time the concepts and codes that were current in 1939 and 1940[69,31] assumed local unloading of initially plastically deformed fibers for columns (Engesser-Kármán theory). The assumed stress-strain diagram for structrual steel was then derived from the assumed buckling stress-slenderness relation for columns. To find the buckling stress for plates, however, the writer[67,68,69,31] chose his formulas in such a way that the resulting buckling stresses were practically the same as would be the case for complete plastic behavior in columns and plates.[75] This was substantiated by additional data published in 1947.[76] For a given stress-strain diagram (as in the case of aluminum) these complications do not arise, so that the writer's computation, for plates that were completely plastic was straight forward and direct.[70,77] In 1947, F. R. Shanley[78] showed that even an initially straight column should be assumed to remain entirely plastic with incipient buckling. Mr. Shanley's principle leads to the same conclusion for a plate.

The buckling stresses and wave lengths of buckles, predicted in 1938 by the writer's theory of plastic buckling,[67] then were confirmed by the writer in 1947[70,77] and in 1949,[79] as well as by experiments reported in 1946[80] by C. P. Kollbrunner. A summary of tests by NACA, which serves to confirm the writer's work, was published in 1948.[81] At first, the writer's theory received scant attention in the United States, but it was observed by A. A. Ilyushin[82] in Russia. The latter tried to simplify it by assuming Poisson's ratio for the elastic deformations equal to 0.5; but on the other hand, he assumed local unloading of initially plastically deformed regions and tried to find the exact solution for the boundary surface between the elastic and plastic region. However, he failed in this purpose by starting from a wrong assumption.[79,83] Elbridge Z. Stowell[32] changed Mr. Ilyushin's analysis by assuming the plate to remain entirely plastic, analogous to Mr. Shanley's principle for columns.[78] Hence, the only difference between Mr. Stowell's computations and those based on the writer's theory is that Mr. Stowell assumes Poisson's

[75] "Theory of the Plastic Stability of Thin Plates," by P. P. Bijlaard, *Publications*, International Assn. for Bridge and Structural Eng., Vol. 6, 1940–1941 (see footnote 10).

[76] "Some Contributions to the Theory of Elastic and Plastic Stability," by P. P. Bijlaard, *ibid.*, Vol. 8, 1947, p. 28.

[77] "On the Plastic Stability of Thin Plates and Shells," by P. P. Bijlaard *Proceedings*, Royal Netherlands Academy of Sciences, Amsterdam, Holland, Vol. 50, September, 1947, pp. 765–775.

[78] "Inelastic Column Theory," by F. R. Shanley, *Journal of the Aeronautical Sciences*, Vol. 14, No. 5, 1947, pp. 261–268.

[79] "Theory and Tests on the Plastic Stability of Plates and Shells," by P. P. Bijlaard, *ibid.*, Vol. 16, No. 9, 1949, pp. 529–541.

[80] "Das Ausbeulen der auf Einseitigen, Gleichmässig Verteilten Druck Beanspruchten Platten im-Elastischen und Plastischen Bereich," by C. F. Kollbrunner, *Mitteilungen a.d. Institut f. Baustatik a.d. E.T.H.*, Zurich, Switzerland, No. 17, 1946.

[81] "Prediction and Reduction to Minimum Properties of Plate Compressive Curves," by E. H. Schuette and J. C. McDonald, *Journal of the Aeronautical Sciences*, Vol. 15, No. 1, 1948, pp. 23–27.

[82] "Stability of Plates and Shells Beyond the Proportional Limit," by A. A. Ilyushin, *Technical Memorandum No. 1116*, NACA, October, 1947 (Translation from Prikladnaya Matematika i Mekhanika, Vol. 8, 1944).

[83] "Buckling Stresses of Box-Beams under Pure Bending," by Sigge F. Eggwertz, *Meddelande No. 33*, Flygtekniska Försöksanstalten (FFA), Stockholm, Sweden, 1950.

[32] "A Unified Theory of Plastic Buckling of Columns and Plates," by Elbridge Z. Stowell, *Report No. 898*, NACA, 1948.

ratio for the elastic deformations equal to 0.5, instead of its actual value of about 0.3. It seems useful to state this relation between the foregoing methods of computation, since they are sometimes presented and thought of as entirely different theories.

Applications of his theory to aircraft structures were given by the writer in 1949,[79] 1950,[84,85] and 1951.[86] In a number of instances, the writer[67,79,87,88] has explained why, for plastic buckling of plates, the so-called deformation theory rather than the flow theory of plasticity should be used. The same was contended by S. B. Batdorf.[59] All of these views have been substantiated by experiments.[79,88] The writer's theory was especially mentioned as a sound basis for design in the conclusions, accepted by the Third Congress of the International Association for Bridge and Structural Engineering at Liege, Belgium, September, 1948.[89]

ELBRIDGE Z. STOWELL,[90] GEORGE J. HEIMERL,[91] CHARLES LIBOVE,[92] J. M. ASCE, AND EUGENE E. LUNDQUIST,[93] A. M. ASCE.—The contribution by Mr. Bijlaard is welcome. Much of his work has appeared in foreign journals and has not been readily available. The first time that his work came to the writers' attention was in the paper by Mr. Ilyushin.[82]

A final acknowledgment is in order to the Column Research Council. It was at their suggestion that this paper was prepared and presented to the Society.

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[84] "Analysis of the Elastic and Plastic Stability of Sandwich Plates by the Method of Split Rigidities, I–II," by P. P. Bijlaard, *Preprint Institute of Aeronautical Sciences No. 259*, January, 1950.

[85] "Stability of Alclad Plates," by P. P. Bijlaard, *Journal of the Aeronautical Sciences*, Vol. 17, No. 4, 1950, p. 252.

[86] *Ibid.*, Vol. 18, 1951, No. 5, pp. 339–349 and No. 12, pp. 790–796, 829.

[87] "On the Plastic Buckling of Plates," by P. P. Bijlaard, *ibid.*, Vol. 17, No. 11, 1950, pp. 742–743.

[88] "On the Plastic Buckling of Plates According to the Flow Theory," by P. P. Bijlaard, *ibid.*, No. 12, 1950, pp. 810–811.

[59] "Theories of Plastic Buckling," by S. B. Batdorf, *ibid.*, Vol. 16, No. 7, July, 1949, p. 405.

[89] "Conclusions and Suggestions," Final Report of the Third Congress of the International Assn. for Bridge and Structural Eng., Liege, Belgium, September, 1948, pp. 113–115.

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